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LAGRANGE'S EQUATION IN THE CALCULUS OF VARIATIONS, AND THE EXTENSION OF A THEOREM OF ERDMANN.

BY J. K. WHITTEMORE.

1. If y is a function of x which satisfies certain conditions of continuity and makes the integral

$$I = \int_{x_0}^{x_1} F(x, y, y')\,dx$$

a minimum, then, as is well known, the following relation must be satisfied:*

$$(1) \qquad \int_{x_0}^{x_1} (\eta F_y + \eta' F_{y'})\,dx = 0, \qquad \left(F_y = \frac{\partial F}{\partial y}, \text{ etc.}\right)$$

where η denotes an arbitrary variation of y.

It is the purpose of this paper to deduce from (1) the differential equation, known sometimes as Lagrange's Necessary Condition, with the fewest possible additional assumptions concerning y and F, and also to prove a general theorem regarding the continuity of y' which has been shown by Erdmann to hold in certain special cases.† This condition is

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* For a proof of this theorem *cf.*, for example, Professor Osgood's paper, this number of the ANNALS, p. 107.

In the present paper it is assumed:

(*a*) that the function $F(x, y, p)$, regarded as a function of the three independent variables (x, y, p), is, together with its first and those of its second partial derivatives which present themselves, continuous throughout the region

$$R: \qquad x_0 \leqq x \leqq x_1 \qquad L \leqq y \leqq M, \qquad L' \leqq p \leqq M';$$

(*b*) that the function y is continuous throughout the interval $x_0 \leqq x \leqq x_1$ and has a continuous first derivative at all points of this interval with the exception, at most, of a finite number of points; and that y' remains finite throughout the interval.

† Erdmann, *Crelle*, vol. 82 (1876), p. 21. A relation is there established from which it follows in many cases that the graph of a function y which makes the integral I a minimum cannot have a vertex. Weierstrass extended Erdmann's theorem to the case that x and y are expressed as functions of a parameter t, obtaining the equations

$$\left[F_{x'}\right]_{t_2 - 0} = \left[F_{x'}\right]_{t_2 + 0}, \qquad \left[F_{y'}\right]_{t_2 - 0} = \left[F_{y'}\right]_{t_2 + 0}.$$

Cf. Kneser, *Lehrbuch der Variationsrechnung* p. 172.

$$F_y - \frac{dF_{y'}}{dx} = 0 \tag{2}$$

Let us consider first the usual method of obtaining the differential equation (2) from (1). If we integrate by parts the second term of the integrand, we have:

$$\int_{x_0}^{x_1} y' F_{y'} dx = \Big[\eta F_{y'}\Big]_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta \frac{dF_{y'}}{dx} dx.$$

Here the term $\Big[\eta F_{y'}\Big]_{x_0}^{x_1}$ vanishes, if we assume $\eta(x_0) = \eta(x_1) = 0$, and therefore

$$\int_{x_0}^{x_1} \eta' F_{y'} dx = -\int_{x_0}^{x_1} \eta \frac{dF_{y'}}{dx} dx.$$

The necessary condition (1) thus becomes

$$\int_{x_0}^{x_1} \eta \left(F_y - \frac{dF_{y'}}{dx}\right) dx = 0.$$

From this equation it may be proved — and in this proof we are not at present interested — that

$$F_y - \frac{dF_{y'}}{dx} = 0.$$

We observe that the integration by parts here employed assumes the existence of $dF_{y'}/dx$ and hence, when $F_{y'y'} \neq 0$, of y''. It is just this assumption which we wish to avoid, for every assumption made concerning y restricts the scope of the problem solved. If we assume any property of y, we can know of the function y, found as a solution of the problem, only that it makes the integral I smaller than any other function, y, having the same properties. To remove any assumption concerning y is to enlarge the solution of the problem, — it is adding to the functions which give the integral a greater value than the y found as a solution. For example, consider the problem of finding the shortest line connecting two points of a plane. We define the length of the arc of the curve $y = f(x)$ between two of its points, (x_0, y_0) and (x_1, y_1), as

$$L = \int_{x_0}^{x_1} \sqrt{1 + y'^2}\, dx.$$

If we assume the existence of y'' we find by the Calculus of Variations that the shortest line is the straight line joining the two points. It is however quite conceivable that there should exist a function y, which has no second derivative y'', which would make the integral L smaller than the function y which defines the straight line. If we can avoid assuming the existence of y'' this possibility is removed.

The method of transforming the integral (1) which I have given above was, I think, used by all writers on the Calculus of Variations preceding Du Bois Reymond.[*] He showed how we may obtain equation (2) from (1) by assuming the continuity of y' without assuming the existence of y''. In this paper I shall show, following to a certain extent the method of Du Bois Reymond, as it was presented by Hilbert in a course of lectures on the Calculus of Variations delivered at Göttingen in the summer of 1899, that it is not necessary to assume even the continuity of y' at all points of the interval (x_0, x_1).

I assume for simplicity that y' has only a finite number of discontinuities; but the method employed applies without modification to the case that y' has an infinite number of discontinuities in the interval, the content of this set of points being zero. As regards the function F, I assume

(a), for the proof of the existence of y'' at a point in whose neighborhood y' is continuous, that $F_{y'y'}(x, y, y') \neq 0$ at this point;

(b), for the proof of the continuity of y' at a given point, that $F_{y'y'}(x, y, Y') \neq 0$, where Y' has any value between (and inclusive of) the limits within which y' lies in the neighborhood of this point.

It is interesting to note that this latter assumption is at the same time, provided a field exists, Weierstrass's sufficient condition for a minimum for strong variations.

2. To avoid the necessity of assuming the existence of the derivative $dF_{y'}/dx$, and hence of y'', we integrate by parts the first instead of the second term of the integrand in (1). We have

$$\int_{x_0}^{x_1} \eta F_y \, dx = \left[\eta \int_{x_0}^{x} F_y \, dx\right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \left(\eta' \int_{x_0}^{x} F_y \, dx\right) dx.$$

We shall, by a particular choice of the variation η, be able to show that the function

$$F_{y'} - \int_{x_0}^{x} F_y \, dx = P(x)$$

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[*] *Math. Annalen*, vol. 15 (1879), p. 564.

is a constant. In deducing a necessary condition for a minimum we may evidently use a particular variation. From the fact that $P(x)$ is constant we shall prove the continuity of y' and the existence and continuity of y''.

The points at which y' is discontinuous or does not exist are by hypothesis finite in number. We may, then, choose two intervals α_1 to β_1 and α_2 to β_2 so that

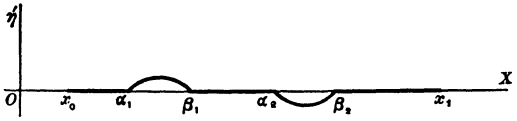
$$x_0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < x_1,$$
$$\beta_1 - \alpha_1 = \beta_2 - \alpha_2,$$

and further so that when

$$\alpha_1 \leqq x \leqq \beta_1 \qquad \text{or} \qquad \alpha_2 \leqq x \leqq \beta_2$$

$y'(x)$ is continuous.

We will now choose η', the derivative of the variation η, as follows, where ϵ is a positive constant.



$$\begin{array}{lcr}
\eta' = 0 & \text{when} & x_0 \leqq x \leqq \alpha_1 \\
\eta' = \epsilon(x - \alpha_1)(\beta_1 - x) & \text{``} & \alpha_1 < x \leqq \beta_1 \\
\eta' = 0 & \text{``} & \beta_1 < x \leqq \alpha_2 \\
\eta' = -\,\epsilon(x - \alpha_2)(\beta_2 - x) & \text{``} & \alpha_2 < x \leqq \beta_2 \\
\eta' = 0 & \text{``} & \beta_2 < x \leqq x_1
\end{array}$$

We see that η' is continuous; the derivative of η' is however not continuous. That is immaterial for our purpose.

The function η, whose derivative is thus defined, which vanishes for $x = x_0$ is given by the formula:

$$\eta = \int_{x_0}^{x} \eta' dx.$$

Finally, $\eta(x_1) = 0$. Since ϵ is arbitrary, it may be chosen so small that η will be a weak variation. This is desirable, since a necessary condition that holds even when only weak variations are considered is a broader condition than one that does not hold unless strong variations are admitted.

We now transform (1), integrating by parts the term $\eta F'_y$. We have

$$\int_{x_0}^{x_1} \eta F_y dx = \left[\eta \int_{x_0}^{x} F_y dx\right]_{x_0}^{x_1} - \int_{x_0}^{x_1}\left(\eta' \int_{x_0}^{x} F_y dx\right) dx$$
$$= - \int_{x_0}^{x_1}\left(\eta' \int_{x_0}^{x} F_y dx\right) dx.$$

Equation (1) thus becomes

$$\int_{x_0}^{x_1} \eta' \left(F_{y'} - \int_{x_0}^{x} F_y dx\right) dx = 0$$

or

$$\int_{x_0}^{x_1} \eta' P(x) dx = 0.$$

Since $\eta' = 0$ except when x is between α_1 and β_1 or between α_2 and β_2, we may write

$$\int_{\alpha_1}^{\beta_1} \epsilon(x - \alpha_1)(\beta_1 - x) P(x) dx - \int_{\alpha_2}^{\beta_2} \epsilon(x - \alpha_2)(\beta_2 - x) P(x) dx = 0.$$

In the second integral let us change the variable of integration to y, where

$$y = x + \alpha_1 - \alpha_2.$$

The equation becomes then, if in the second integral we write x instead of y and if we combine the two integrals,

$$(3) \qquad \int_{\alpha_1}^{\beta_1} \epsilon(x - \alpha_1)(\beta_1 - x)[P(x) - P(x + \alpha_2 - \alpha_1)] dx = 0.$$

The factor in brackets is a continuous function of x, for we have supposed y continuous for all values of x between α_1 and β_1 or between α_2 and β_2, and $F_{y'}$ and F_y are continuous functions of x, y, and y'. Now the factor $\epsilon(x - \alpha_1)(\beta_1 - x)$ is never negative within the limits of integration. Hence for some value of x between α_1 and β_1, we must have

$$P(x) - P(x + \alpha_2 - \alpha_1) = 0$$

since this function of x must change sign, and, being continuous, must vanish in the interval. Let this value of x be

$$x = \alpha_1 + \theta(\beta_1 - \alpha_1), \qquad 0 \leqq \theta \leqq 1.$$
$$P[\alpha_1 + \theta(\beta_1 - \alpha_1)] = P[\alpha_2 + \theta(\beta_1 - \alpha_1)].$$

We may now let $\beta_1 - \alpha_1$ approach zero, and since in the interval considered $P(x)$ is continuous,

$$P(\alpha_1) = P(\alpha_2),$$

where α_1 and α_2 are any two points of the interval x_0 to x_1 where y' is continuous.

Thus it appears that the function $P(x)$ has one and the same constant value, C, throughout the whole interval from x_0 to x_1 with the exception at most of a finite number of points, namely those at which we have not assumed the function y' to be continuous. Let a_3 be one of these points. Introduce a function $\omega(x)$ which shall be equal to y' at all points at which y' is continuous, and at the point a_3 shall have a value presently to be defined. We shall show that this value may be chosen so that the function

$$P(x) = F_{y'}(x, y, \omega) - \int_{x_0}^{x} F_y(x, y, y')dx$$

will be continuous at $x = a_3$.

The function

$$\phi(x, \omega) = F_{y'}(x, y, \omega) - \int_{x_0}^{x} F_y(x, y, y')dx$$

regarded as a function of the independent variables, x and ω, is continuous in these variables, and it has the value C when ω is the function of x above defined and $x \neq a_3$.

We may choose a positive number δ so small that, when x satisfies the inequality

$$a_3 < x \leqq a_3 + \delta,$$

y' is continuous. Now, as x approaches a_3, $\omega(x)$ may conceivably approach no limit. Whether it does or not, there is certainly, since y' is restricted to finite limits, at least one value, Ω, such that there are points in every neighborhood of a_3 for which the corresponding values y' differ from Ω by less than any preassigned positive value. Let any such value Ω be chosen as the definition of $\omega(x)$ when $x = a_3$: $\omega(a_3) = \Omega$. Now let x approach a_3 passing through only such values of x that $\omega(x)$ approaches $\omega(a_3)$. Since $\phi(x, \omega)$ is a continuous function of x and ω, we have,

$$\phi(a_3, \Omega) = \lim_{\substack{x \doteq a_3 \\ \omega \doteq \Omega}} \phi(x, \omega) = C$$

Suppose $\omega(x)$ is in this way defined at every point for which y' is discontinuous or not defined. Then $\omega(x)$ is defined at all points of the interval x_0 to x_1, and we have obtained the result that, *for every value of* x,

$$P(x) = \phi(x, \omega(x)) = C.$$

I will now prove that $\omega(x)$ is continuous. We have

$$\phi(x + \Delta x, \omega(x + \Delta x)) - \phi(x, \omega(x)) = 0$$

Putting for ϕ in this equation its value, and using the mean value theorem, we have

$$\Delta\omega\, F_{y'y'}(x+\Delta x,\ y+\Delta y,\ \omega+\theta\Delta\omega)+\Delta y\, F_{yy}\,(x+\Delta x,\ y+\theta_1\Delta y,\ \omega)$$
$$+\ \Delta x F_{xy'}(x+\theta_2\Delta x,\ y,\ \omega)-\int_x^{x+\Delta x} F_y\,dx \;=\; 0$$

or, solving for $\Delta\omega$,

$$(4)\quad \Delta\omega = \frac{\displaystyle\int_x^{x+\Delta x} F_y\,dx - \Delta F_{xy'}(x+\theta_2\Delta x,\ y,\ \omega) - \Delta y F_{yy'}(x+\Delta x,\ y+\theta_1\Delta y,\ \omega)}{F_{y'y'}(x+\Delta x,\ y+\Delta y,\ \omega+\theta\Delta\omega)}$$

If now we let Δx approach zero, the numerator of the fraction approaches zero, and the denominator does not, since by hypothesis

$$F_{y'y'}(x,\ y,\ y') \neq 0 \qquad L' \leqq y' \leqq M'$$

Hence $\lim\limits_{\Delta x = 0} \Delta\omega = 0$, and $\omega(x)$ is a continuous function of x.

It can now be shown that y' exists at the point $x = a_3$, and that the function $y'(x)$ is at this point continuous. For since y is a continuous function in the interval $a_3 \leqq x \leqq a_3 + \Delta x$(or, when $\Delta x < 0$, in the interval $a_3 + \Delta x \leqq x \leqq a_3$), and since y has a derivative y' at all internal points of this interval, we may apply to the function y the law of the mean. We have

$$\frac{y(a_3+\Delta x) - y(a_3)}{\Delta x} = y'(a_3 + \theta\,\Delta x).$$

Since we have shown that $y'(a_3 + \theta\,\Delta\,x)$ approaches a limit as Δx approaches zero, the existence and continuity of y' at $x = a_3$ follow at once.

To prove the existence and continuity of y'', divide equation (4) through by Δx and notice that $y' = \omega(x)$ for all values of x in the interval (x_0, x_1). Now let Δx approach 0 and we have

$$\lim_{\Delta x = 0} \frac{\Delta y'}{\Delta x} = \frac{F_y - F_{xy'} - y'F_{yy'}}{F_{y'y'}}$$

Finally we may notice that, if the existence of the third partial derivatives of F is assumed, we may in the same way prove that y''' exists, and similarly for the higher derivatives.

Harvard University,
Cambridge, Massachusetts.